Variable-coefficient wave equations with exact spreading solutions

1993 J. Phys. A: Math. Gen. 26 5491

(http://iopscience.iop.org/0305-4470/26/20/032)

J. Phys. A: Math. Gen. 26 (1993) 5491–5504. Printed in the UK

Variable-coefficient wave equations with exact spreading solutions

W E Couch and R J Torrence

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4

Received 14 December 1992

Abstract. We discuss a family of variable-coefficient linear wave equations that are exactly solved by a generalization of progressing waves that spread out as they propagate. This can be viewed as a generalization to some variable-coefficient equations of the dispersion theory method that is standard for constant coefficient equations. The method is illustrated by solving some physically interesting equations.

1. Introduction

In this paper we shall solve a significant family of variable-coefficient linear wave equations by methods that can be viewed as a generalization of the dispersion theory techniques standard for constant coefficient linear wave equations. The solutions will be a natural generalization of progressing waves and will be exact in a useful sense, and they will in general be dispersive in that they will spread out as they propagate. Our results will be obtained by applying a new theorem regarding wave equations that may also have other interesting consequences.

The ordinary wave equation in one space dimension

$$(\partial_t^2 - \partial_x^2)\phi = 0 \tag{1.1}$$

is satisfied by the waves

$$\phi = e^{i(\omega t - kx)} \tag{1.2}$$

when

$$\omega^2 - k^2 = 0\,. \tag{1.3}$$

All of these waves have the same phase velocity, $\omega/k = 1$, up to sign, and it follows that if we form two solutions depending on arbitrary functions $g_\pm(k)$ by writing

$$\phi_\pm = \int_{-\infty}^{\infty} dk\; g_\pm(k) e^{i(\omega t \mp kx)} \tag{1.4}$$

they will move in the positive and negative x directions, respectively, and they will propagate without spreading out. Thus equation (1.1), and the waves (1.4), are commonly referred to in the literature as being 'non-dispersive'. In this paper we shall call any wave in one

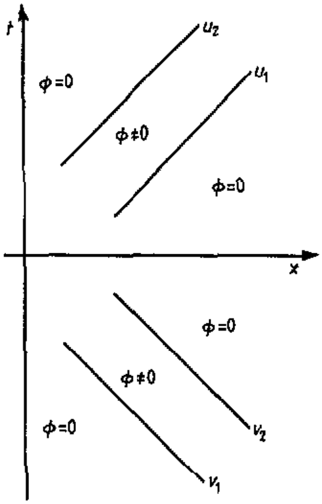

Figure 1. The propagation of non-spreading waves.

space dimension a *non-spreading wave* when, if its support at some time t_0 falls between a pair of retarded (advanced) characteristics u_1, u_2 (v_1, v_2), where $t = v + u, x = v - u$, it does so at all times t. We shall also call any wave equation in one space dimension a *non-spreading wave equation* when its general solution can be expressed in terms of such waves. The propagation of non-spreading waves is pictured in figure 1.

Of course the non-dispersive equation (1.1) and waves (1.4) exemplify non-spreading, but we have introduced the new word to describe and denote the general phenomenon because the term 'non-dispersive' is usually applied in the context of constant coefficient wave equations, while in this paper we are interested in wave equations with variable, i.e. spacetime dependent, coefficients.

Non-spreading equations like (1.1) with exact non-spreading wave solutions like (1.4) are a comparitive rarity, as we shall see in section 2, but there is also a class of wave equations that are exactly solvable by standard methods in terms of waves that are not non-spreading. If (1.1) is generalized to

$$(\partial_t^2 - \partial_x^2 + m^2)\phi = 0 \tag{1.5}$$

where m is a constant, (1.2) is a solution when

$$\omega^2 - k^2 - m^2 = 0\,. \tag{1.6}$$

Given (1.6), waves with different wavenumbers k have different phase velocities $\omega/k = [1 + (m^2/k^2)]^{1/2}$, up to sign, and for this reason the solutions

$$\phi_\pm = \int_{-\infty}^{\infty} \mathrm{d}k\; g_\pm(k)\mathrm{e}^{\mathrm{i}[\omega(k)t \mp kx]} \tag{1.7}$$

spread out as they propagate. Thus the equation (1.5) and its solutions (1.7) are customarily referred to as being 'dispersive', and are examples of what we shall call *spreading* equations and *spreading* solutions, respectively. The further generalization of (1.5) to constant coefficient linear wave equations of any order is handled in a similar fashion to provide solutions analagous to (1.7), which are generally spreading, with the algebraic relation (1.6) generalizing to a polynomial condition

$$\Omega(\omega, k) = 0 \tag{1.8}$$

to be solved for $\omega(k)$. These dispersion theory methods are discussed with care in many places, and an asymptotic analysis of the resulting solutions leads naturally to the important notion of group velocity, $v_g(k) = \mathrm{d}\omega/\mathrm{d}k$, the velocity of propagation of the wavenumber k [1].

There do exist in the literature some attempts to generalize dispersion theory methods to deal with wave equations with spacetime dependent coefficients. The most ambitious work along these lines appears to be Whitham's 'variational approach' [1–3], however it seems to be both more and less than we have in mind here. It is more in the sense that it is even applicable to nonlinear equations, while we shall consider only linear equations. It is less in that when it is specialized to linear equations it yields an *inexact* analysis for waves whose period and wavelength are small compared with the time and distance scales for changes in the wave equation's coefficients, while in this paper we shall present a significant family of variable-coefficient linear wave equations with general solutions that have *exact* representations in the same sense that (1.7) exactly solves (1.5).

Our approach will be to combine a method of exact solvability by means of non-spreading waves of a relatively large family of non-spreading equations, with the dispersion theory method just illustrated for constant coefficient equations, to obtain exact spreading solutions of a class of equations that generalize both of the above families. We first want to find a family of variable-coefficient equations that are analogous to (1.1) in the sense that they have non-spreading solutions, and in section 2 we shall argue that *progressing waves* [4–6] are appropriate, although their defining representation is not a direct generalization of (1.4). We shall also review a procedure [7, 8] for constructing probably all second-order linear wave equations in one spatial dimension whose general solutions can be expressed as the sum of two progressing waves, which ensures an adequate supply of non-spreading waves and equations for our purposes, and we shall give a few special classes of such variable-coefficient wave equations that will be useful later in the paper for illustrative purposes. In section 3 we shall prove two theorems that show how to use solutions to pairs of wave equations of appropriate types to generate solutions to related, more complicated, wave equations. The theorems are then applied to the construction of exact spreading solutions of a relatively wide class of variable-coefficient equations, which is our main result. In the first instance these waves will be given by the action of a 'progressing wave operator', to be defined in section 2, on spreading 'potentials', but one easily obtains an alternative representation that is a natural generalization of (1.7). In section 4 we shall illustrate the method by obtaining a novel representation of the general solution of the massive Klein–Gordon equation in Minkowski spacetime, and the general solution to an equation that governs the development of density perturbations of some physically important cosmological spacetimes.

As far as we know the two theorems of section 3 are new, and lead to what appears to us to be a new and useful generalization of conventional dispersion techniques. Most of our calculations are done in one space dimension, but it will be clear in sections 3 and 4 that these results have immediate implications for higher-dimensional spacetimes as well.

2. Progressing waves

If we rewrite (1.1) in the form

$$\partial^2_{uv}\phi = 0 \tag{2.1}$$

where $t = v + u$, and $x = v - u$, its general solution is obviously a linear combination of

$$\phi_{\pm} = f(t \mp x) \tag{2.2}$$

where f is any sufficiently differentiable function. The propagation properties illustrated in figure 1 are more immediately evident with the representation (2.2), and we shall generalize the notion of non-spreading waves by starting with the latter. It is clear that those propagation properties will be retained if we replace (2.2) by

$$\phi_{\pm} = c^{\pm}(u, v)f(t \mp x) \tag{2.3}$$

where $c^{\pm}(u, v)$ are fixed functions in the sense that they will depend on and be determined by the wave equation to be solved, while $f(t \mp x)$ are any suffiently differentiable functions of one variable. Waves of precisely this form have been discussed in the literature, for example by Courant and Hilbert [5], who referred to them as 'relatively undistorted progressing waves', by Friedlander [6], who called them 'simple progressing waves',and more recently by Eckhoff [9], for whom they are 'dispersion free waves', and Hillion [10], who labelled them 'dispersionless waves'. Whatever they are called, it is clear that the waves (2.3) have the non-spreading propagation property depicted in figure 1, despite the *distortion* during propagation caused by the coefficient functions $c^{\pm}(u, v)$. The simple observation with which we begin is that if we follow [5, 6] and define a *progressing wave of order N* to be a function of the form

$$\phi_{\pm} = \sum_{n=0}^{N} c_n^{\pm}(u, v) f_{N-n}(t \mp x) \tag{2.4}$$

where the $c_n^{\pm}(u, v)$ are fixed functions, in the above sense, and

$$f_n(z) = \mathrm{d}f_{n-1}(z)/\mathrm{d}z \tag{2.5}$$

with $f_0(z)$ any sufficiently differentiable function of one variable, then (2.4) also has the non-spreading propagation property of (2.3), the latter being subsumed under (2.4) as progressing waves of order 0. The special form of the functions $f_n(z)$ given by (2.5) anticipates the eventual requirement that we shall want (2.4) to satisfy some linear wave equation, and is relevant to non-spreading as it restricts the support of ϕ to the support of f_0. The specialization of (2.4) obtained by assuming $f_0(z) = \mathrm{e}^{\mathrm{i}z}$ leads to the generalizations of (1.2) and of (1.4) appropriate to non-spreading variable-coefficient equations. That the waves (2.4) have the same simple propagation property as (2.3) is obvious, but seems not to have been exploited before as we intend to do here.

A systematic search for wave equations with non-spreading general solutions was undertaken by Kundt and Newman [7] in 1968. A brief survey of their results, as extended in [8], is appropriate here. It is not difficult to show that every second-order linear wave equation in one space dimension can be put into either one of the two forms $(\partial_v a \partial_u - b)\phi = 0$

and $(\partial_u c \partial_v - d)\psi = 0$, where a, b, c and d depend in general on u and v. If we define $j_0 = a$, $j_1 = b$, $\phi_0 = \phi$, and inductively define $\{j_k\}_{k\in Z}$, $\{\phi_k\}_{k\in Z}$ by

$$j_{k+1}/j_k = j_k/j_{k-1} - \partial_u[(\partial_v j_k)/j_k] \qquad k \in Z \tag{2.6}$$

$$j_{k+1}\phi_{k+1} = j_k \partial_u \phi_k \qquad k \in Z \tag{2.7}$$

we obtain a countable set of equations

$$(\partial_v j_k \partial_u - j_{k+1})\phi_k = 0 \qquad k \in Z \tag{2.8}$$

while if we define $l_0 = c$, $l_{-1} = d$, $\psi_0 = \psi$, and inductively define $\{l_k\}_{k\in Z}$, $\{\psi_k\}_{k\in Z}$ by

$$l_{k-1}/l_k = l_k/l_{k+1} - \partial_v[(\partial_u l_k)/l_k] \qquad k \in Z \tag{2.9}$$

$$l_{k-1}\psi_{k-1} = l_k \partial_v \psi_k \qquad k \in Z \tag{2.10}$$

we obtain a second countable set

$$(\partial_u l_k \partial_v - l_{k-1})\psi_k = 0 \qquad k \in Z\,. \tag{2.11}$$

It is easy to confirm that for all $k \in Z$ the kth equation in (2.8), corresponding to the coefficients j_k, j_{k+1}, and the kth equation in (2.11), corresponding to the coefficients l_k, l_{k-1}, are equivalent with

$$j_k l_k = 1 \qquad \phi_k = l_k \psi_k \qquad k \in Z\,. \tag{2.12}$$

It can also be seen that all of the equations in the combined sets (2.8), (2.11) are equivalent in the sense that a solution of any one equation generates a solution of every other equation in both sets. Thus a solution ϕ_{k_0} to the k_0 equation in (2.8) immediately gives solutions to the kth equation, $k > k_0$, in (2.8) by repeated applications of (2.7). But it also gives a solution to the k_0 equation in (2.11) through (2.12), which generates solutions to the kth equation, $k < k_0$, in (2.11) by repeated applications of (2.10). Then (2.12) gives us solutions to the rest of (2.8) and (2.11).

The implication of these results for progressing waves, and thus for non-spreading waves, is seen as follows. Given a wave equation in the form

$$(\partial_v j_0 \partial_u - j_1)\phi_0 = 0 \tag{2.13}$$

we say the 'substitution sequence' $\{j_k\}_{k\in Z}$ is double terminating when $j_{N_1+1} = 0$ and $l_{-N_0-1} = 1/j_{-N_0-1} = 0$ for some $N_1 \geqslant 0$, $-N_0 \leqslant 0$. But it follows, using (2.6)–(2.11), that (2.13) is solved by $\phi_0 = \phi_- + l_0\psi_+$ where

$$\psi_+ = (l_{-1}^{-1}\partial_u l_{-1}) \cdots (l_{-N_0}^{-1}\partial_u l_{-N_0}) f(t-x) \tag{2.14}$$

and

$$\phi_- = (j_1^{-1}\partial_v j_1) \cdots (j_{N_1}^{-1}\partial_v j_{N_1}) f(t+x) \tag{2.15}$$

respectively, where $f(z)$ is any sufficiently differentiable function of one variable. However it is obvious that if one carries out the differentiations in (2.14) and (2.15) and groups the resulting terms according to the order of differentiation of the functions $f(t \mp x)$, the result will be progressing waves of order N_0 and N_1, respectively. General formulas for the coefficients $c_n^+(u,v)$ and $c_n^-(u,v)$ of (2.4) can be written in terms of the $\{l_k\}$ and their iterated u derivatives, and the $\{j_k\}$ and their iterated v derivatives, respectively, but they are not particularly useful here. The derivation of these solutions, (2.14) and (2.15), of (2.13) depended on the double termination conditions, and one readily observes that this amounts, through (2.6) and (2.9), to a pair of ostensibly intractable nonlinear differential conditions of order $2N_1$ and $2N_0$, respectively, on the coefficients j_0 and j_1 of the original equation (2.12). However, it was observed in [8] that to satisfy these conditions is equivalent to finding a solution of the finite Toda lattice [11] with free ends, of length N_0+N_1+1, and since a general solution to that integrable dynamical system is available [12], all double terminating substitution sequences, and thus probably all wave equations (2.13) that are non-spreading, are, in principle, known. The general construction is complicated, and for the details the reader is referred to [8, 12].

In the next section we shall want the specialization of the material just reviewed to the cases

$$j_0 = 1 \qquad j_{-k} = j_k^{-1} \qquad j_1(u,v) = j_1^{\mathrm{S}}(u-v) = j_1^{\mathrm{S}}(x) \tag{2.16}$$

and

$$j_0 = 1 \qquad j_{-k} = j_k^{-1} \qquad j_1(u,v) = j_1^{\mathrm{T}}(v+u) = j_1^{\mathrm{T}}(t) \tag{2.17}$$

where in each case the second equation is a consequence [7, 8] of $j_0 = 1$, and implies in the case of double termination that $N_0 = N_1 = N$. Using the fact that $\psi_0 = \phi_0$ when $j_0 = 1$, and $\partial_u = \partial_t - \partial_x$ and $\partial_v = \partial_t + \partial_x$, it is now straightforward to derive that (2.14) and (2.15) can be combined into

$$\phi_\pm^{\mathrm{T}} = \sum_{n=0}^{N} c_n^\pm(t) f_{N-n}(t \mp x) \tag{2.18}$$

for $j_1 = j_1^{\mathrm{T}}(t)$, where $f_n(t \mp x) = \partial_t f_{n-1}(t \mp x)$, and

$$\phi_\pm^{\mathrm{S}} = \sum_{n=0}^{N} c_n^\pm(x) f_{N-n}(t \mp x) \tag{2.19}$$

for $j_1 = j_1^{\mathrm{S}}(x)$, where $f_n(t \mp x) = \partial_x f_{n-1}(t \mp x)$. The functional form of the coefficients $c_n^\pm(z)$ is determined by the functional forms $j_1^{\mathrm{T}}(z)$ or $j_1^{\mathrm{S}}(z)$ and is given by the same formula for $z = t$ and $z = x$, and for this reason the $c_n^\pm(z)$ can simply be written as $c_n(z)$ in these two cases. If we now define the *progressing wave operator*

$$\mathcal{J}_z : \{f(z)\} \longrightarrow \{f(z)\} : f(z) \longmapsto \sum_{n=0}^{N} c_n(z) \partial_z^{N-n} f(z) \tag{2.20}$$

then, putting $f_0(z) = f(z)$, (2.18) and (2.19) take the succinct forms

$$\phi_\pm^{\mathrm{T}} = \mathcal{J}_t f(t \mp x) \tag{2.21}$$

and

$$\phi_\pm^{\mathrm{S}} = \mathcal{J}_x f(t \mp x) \tag{2.22}$$

respectively. There are two aspects of the definition of $\mathcal{J}_z$ that should be emphasized: (i) If a substitution sequence $\{j_k(z)\}$ does not double terminate then the coefficients $\{c_n(z)\}$, and thus the progressing wave operator $\mathcal{J}_z$, are not defined, and (ii) different double terminating substitution sequences imply different sets of coefficients $c_n(z)$, and thus different progressing wave operators $\mathcal{J}_z$, and it is sometimes expedient to stress this by writing $\mathcal{J}_z(\{j_k\})$. It is obvious that the more general progressing waves $\phi_\pm$ and $\psi_\pm$ of (2.14) and (2.15) could also be written as an appropriate *pair* of operators on $f(t \mp x)$, but it is precisely the specialized forms (2.21) and (2.22), defined in terms of the single operator $\mathcal{J}_z$ defined by (2.20), that will play a key role in the theorems to be proved in the next section.

We shall conclude this section by recording a few families of double terminating substitution sequences of the types (2.16) and (2.17) that will be of use in section 4; these and many others were derived in [13], although that paper made no direct use of the concept of progressing wave. Three families with x dependence are generated from $j_0 = 1$ and

$$j_1^{\mathrm{S}}(x) = \begin{cases} -l(l+1)/x^2 \\ -l(l+1)d^2/\sin^2 dx \\ -l(l+1)d^2/\sinh^2 dx \end{cases} \tag{2.23}$$

where $l \in Z^+$ and d is an arbitrary constant, and the corresponding sequences, of total length $2l+1$, are given by

$$j_k^{\mathrm{S}}(x) = \begin{cases} (-1)^k x^{-2k} \prod_{i=1}^{k}[l(l+1)-(i-1)i] \\ (-1)^k (d^{-1}\sin dx)^{-2k} \prod_{i=1}^{k}[l(l+1)-(i-1)i] & 1 \leqslant k \leqslant l \\ (-1)^k (d^{-1}\sinh dx)^{-2k} \prod_{i=1}^{k}[l(l+1)-(i-1)i] \end{cases} \tag{2.24}$$

respectively, with $j_{-k}^{\mathrm{S}}(x) = 1/j_k^{\mathrm{S}}(x)$, and three more with t dependence are generated from $j_0 = 1$ and

$$j_1^{\mathrm{T}}(t) = \begin{cases} l(l+1)/t^2 \\ l(l+1)d^2/\sin^2 dt \\ l(l+1)d^2/\sinh^2 dt \end{cases} \tag{2.25}$$

where $l \in Z^+$ and again d is any constant, and the corresponding sequences, of total length $2l+1$, are given by

$$j_k^{\mathrm{T}}(t) = \begin{cases} t^{-2k} \prod_{i=1}^{k}[l(l+1)-(i-1)i] \\ (d^{-1}\sin dt)^{-2k} \prod_{i=+}^{k}[l(l+1)-(i-1)i] & 1 \leqslant k \leqslant l \\ (d^{-1}\sinh dt)^{-2k} \prod_{i=1}^{k}[l(l+1)-(i-1)i] \end{cases} \tag{2.26}$$

respectively, with $j^{\mathrm{T}}_{-k}(t) = 1/j^{\mathrm{T}}_k(t)$. The corresponding Toda lattice motions are simple and interesting, and were discussed in some detail in [14]. For the first case in each set, i.e. for $j^{\mathrm{S}}_1(x) = -l(l+1)/x^2$ and $j^{\mathrm{T}}_1(t) = l(l+1)/t^2$, one gets [13]

$$c_k(z) = (-2)^{l-k}(l+k)!/k!(l-k)!z^k \tag{2.27}$$

as the $c_n(z)$ are clearly insensitive to the sign of j_1. It follows that

$$[\partial_t^2 - \partial_x^2 + l(l+1)/x^2]\phi = 0 \tag{2.28}$$

is satisfied by the non-spreading waves

$$\phi_\pm = \mathcal{J}_x f(t \mp x) = \sum_{k=0}^{l}[(-2)^{l-k}(l+k)!/k!(l-k)!x^k]\partial_x^{l-k} f(t \mp x) \,. \tag{2.29}$$

In elementary texts the solutions to (2.28) are usually expressed in terms of Bessel functions. If we substiute $\mathrm{e}^{\mathrm{i}(t \mp x)}$ for $f(t \mp x)$ in (2.29) we obtain

$$\begin{aligned} \phi_\pm &= \mathrm{e}^{\mathrm{i}t} e^{\mp\mathrm{i}x} \sum_{k=0}^{l}[(-2)^{l-k}(l+k)!/k!(l-k)!](\mp\mathrm{i})^{l-k}/x^k \\ &= \mathrm{e}^{\mathrm{i}t} e^{\mp\mathrm{i}x} \sum_{k=0}^{l}[(l+k)!(\pm 2\mathrm{i})^{l-k}/k!(l-k)!x^k] \\ &= (\pm 2\mathrm{i})^l \mathrm{e}^{\mathrm{i}t} e^{\mp\mathrm{i}x} \sum_{k=0}^{l}[l+k)!/k!(l-k)!](\pm 2\mathrm{i}x)^{-k} \\ &= (\pm 2\mathrm{i})^l \mathrm{e}^{\mathrm{i}t} e^{\mp\mathrm{i}x} B_l(\pm 1/\mathrm{i}x) \end{aligned} \tag{2.30}$$

where the B_l are *Bessel polynomials of degree l*, which are carefully discussed by Krall and Frink [15]. If we scale the frequency ω and the wavenumber k, with $\omega = k$, into (2.30) by $t \to \omega t$ and $x \to kx$, and ignore the overall factor $(\pm 2\mathrm{i})^l$, our solutions to (2.28) are identical to those given in [15], which are explicitly related there to Bessel functions of order $l + \frac{1}{2}$, and to solutions of the ordinary wave equation in Minkowski spacetime. The generation of the coefficients $c_n(z)$ that appear in the definition (2.20) of $\mathcal{J}$ seems to yield relatively complicated expressions in all other cases. Even with the relatively simple formulas for the $\{j_k\}$ in the other two cases in (2.24) and (2.26), the $c_n(z)$ are given by rational functions in the circular, or hyperbolic, functions, of increasing complexity with increasing n. Of course they are easily calculated for small values of n.

3. The basic theorems

In this section we shall apply the progressing wave operator $\mathcal{J}_z$ to provide exact solutions to a class of variable-coefficient wave equations with spreading solutions, as promised. The result depends on the following theorem.

Theorem 1. If the pair $j_0 = 1$ and $j_1^{\mathrm{T}}(t)$ generate a double terminating substitution sequence, in which case

$$\left[\partial_t^2 - \partial_x^2 - j_1^{\mathrm{T}}(t)\right] \mathcal{J}_t f(t \mp x) = 0 \tag{3.1}$$

holds, and if

$$\left[\partial_t^2 - \partial_x^2 - h(x)\right] p(t, x) = 0 \tag{3.2}$$

then

$$\left[\partial_t^2 - \partial_x^2 - h(x) - j_1^{\mathrm{T}}(t)\right] \mathcal{J}_t p(t, x) = 0\,. \tag{3.3}$$

Proof. It follows from (3.1) and the fact that $\mathcal{J}_t$ and ∂_x^2 commute that

$$\left[\mathcal{J}_t \partial_x^2 - \partial_t^2 \mathcal{J}_t + j_1^{\mathrm{T}}(t) \mathcal{J}_t\right] f(t \mp x) = 0 \tag{3.4}$$

and further, since $\partial_x^2 f(t \mp x) = \partial_t^2 f(t \mp x)$, that

$$\left[\mathcal{J}_t \partial_t^2 - \partial_t^2 \mathcal{J}_t + j_1^{\mathrm{T}}(t) \mathcal{J}_t\right] f(t \mp x) = 0 \tag{3.5}$$

where f is any sufficiently differentiable function. But as there is no x dependence in the operator in (3.5), it is appropriate to rewrite the latter as

$$\left[\mathcal{J}_t \partial_t^2 - \partial_t^2 \mathcal{J}_t + j_1^{\mathrm{T}}(t) \mathcal{J}_t\right] f(t) = 0\,. \tag{3.6}$$

If one substitutes the definitions of $j_1^{\mathrm{T}}(t)$ and $\mathcal{J}_t(\{j_k^{\mathrm{T}}(t)\})$ in (3.6) the result is a finite series of the form $\sum_n \mathrm{d}_n f^{(n)}(t) = 0$, and the independence of the derivatives of $f(t)$ shows that we must have $d_n = 0$ for all n. Thus (3.5) should be read as an identity

$$\mathcal{J}_t (\mathrm{d}/\mathrm{d}t)^2 - (\mathrm{d}/\mathrm{d}t)^2 \mathcal{J}_t + j_1^{\mathrm{T}}(t) \mathcal{J}_t = 0 \tag{3.7}$$

where (3.7) can be applied to any sufficiently differentiable function of t, and the presence of other parameters, such as x in (3.5), is irrelevant. Now we have

$$\left[\partial_t^2 - \partial_x^2 - h(x) - j_1^{\mathrm{T}}(t)\right] \mathcal{J}_t p(x, t) = \mathcal{J}_t \left[-\partial_x^2 - h(x)\right] p(t, x) + \left[\partial_t^2 - j_1^{\mathrm{T}}(t)\right] \mathcal{J}_t p(t, x) \tag{3.8}$$

since $\mathcal{J}_t$ and $\partial_x^2 + h(x)$ commute, so using (3.6) we obtain

$$\left[\partial_t^2 - \partial_x^2 - h(x) - j_1^{\mathrm{T}}(t)\right] \mathcal{J}_t p(t, x) = \mathcal{J}_t \left[\partial_t^2 - \partial_x^2 - h(x)\right] p(t, x) = 0 \tag{3.9}$$

where the very last equality is a consequence of (3.2). This establishes (3.3), the desired result. □

Corollary. There is an analogous implication where we replace $j_1^{\mathrm{T}}(t)$ by $g(t)$, $h(x)$ by $j_1^{\mathrm{S}}(x)$, and $\mathcal{J}_t(\{j_k^{\mathrm{T}}\})$ by $\mathcal{J}_x(\{j_k^{\mathrm{S}}\})$.

The function $p(t, x)$ is *not* provided by the theorem, but must be found by solving (3.2) by some means. One case where this is possible, and where the theorem provides a useful exact general solution of (3.4), is when $g(t) = j_1^{\mathrm{T}}(t)$, i.e. when $j_1(t, x) = j_1^{\mathrm{S}}(x) + j_1^{\mathrm{T}}(t)$, with $j_1^{\mathrm{S}}(x)$ and $j_1^{\mathrm{T}}(t)$ *each* generating their own double terminating substitution sequences, $\{j_k^{\mathrm{S}}(x)\}, \{j_k^{\mathrm{T}}(t)\}$, leading to operators $\mathcal{J}_x(\{j_k^{\mathrm{S}}(x)\}), \mathcal{J}_t(\{j_k^{\mathrm{T}}(t)\})$ defined by (2.20) in terms of the calculable coefficients $\{c_k^{\mathrm{S}}(x)\}, \{c_k^{\mathrm{T}}(t)\}$, respectively. As an example the equation

$$\left[\partial_t^2 - \partial_x^2 + \frac{l(l+1)}{x^2} - \frac{q(q+1)}{t^2}\right]\phi = 0 \qquad l, q \in Z^+ \tag{3.10}$$

has the general solution $\phi = \phi_+ + \phi_-$ where

$$\begin{aligned}\phi_\pm &= \mathcal{J}_x\mathcal{J}_t f(t \mp x) \\ &= \sum_{n=0}^{l}\sum_{p=0}^{q}\left[\frac{(-2)^{l+q-n-p}(l+n)!(q+p)!}{n!p!(l-n)!(q-p)!}\right]x^{-n}t^{-p}\partial_x^{l-n}\partial_t^{q-p}f(t \mp x)\,.\end{aligned} \tag{3.11}$$

Of course it is not necessary that $j_1^{\mathrm{S}}(x)$ and $j_1^{\mathrm{T}}(t)$ generate $c_n(x)$ and $c_n(t)$ of the same form as in this example. Since (3.11) is itself a progressing wave, as will be $\mathcal{J}_x\mathcal{J}_t f(t \mp x)$ in general, it follows that if $j_1^{\mathrm{S}}(x)$ and $j_1^{\mathrm{T}}(t)$ each generate double termination substitution sequences, then so does $j_1^{\mathrm{S}}(x) + j_1^{\mathrm{T}}(t)$. This special consequence of theorem 1 was already noted and proved in [16]. If one retraces the path from progressing waves back to finite Toda lattice motions, it is clear that there is a superposition rule whereby any two motions of one-dimensional finite Toda lattices of length l and m, respectively, can be combined to give a motion of a two-dimensional Toda lattice, generally of length $l + m$, but we shall not pursue this here.

A more general situation, and our primary interest in this paper, is that where $h(x)$ is a function such that we can write the exact general solution $p(t, x)$ of (3.2) as the sum of two spreading waves. It follows from theorem 1 that the action of the progressing wave operator $J_t(\{j_k^{\mathrm{T}}(t)\})$ on $p(t, x)$ provides the general solution of equation (3.3), which gives us the exact, spreading, general solution of a linear wave equation that includes the variable-coefficient $j_1^{\mathrm{T}}(t)$. This will certainly be the case when $h(x)$ in (3.2) is a constant, so that the general solution of (3.2), obtained by the usual dispersion theory methods, is given exactly in terms of the spreading waves (1.7). Such cases will result in an obvious generalization of (1.7), where fixed functions of x, t, k and $\omega(k)$ multiply the exponential function. It is easy to see that the usual asymptotic analysis of that form of the solution will yield the usual derivation and formula for the group velocity associated with the dispersion relation $\omega(k)$, as that derivation depends only on the properties of the oscillatory exponential function, while other details of the asymptotic analysis will become more complicated.

The identity (3.5), in the form (3.7), allows us to reformulate the proof of theorem 1 into the proof of the following more general theorem, which may be useful.

Theorem 2. Given

$$[\partial_t^2 - \mathcal{D}_x - j_1^{\mathrm{T}}(t)]\mathcal{J}_t f(t, x) \tag{3.12}$$

and

$$[\partial_t^2 - \mathcal{D}_x - h(x)]p(t, x) = 0 \tag{3.13}$$

where $\mathcal{D}_{\boldsymbol{x}}$ and $h(\boldsymbol{x})$ are an operator and a function of coordinates $x_1, x_2, \ldots x_n$, both commuting with $\mathcal{J}_t$, then

$$[\partial_t^2 - \mathcal{D}_{\boldsymbol{x}} - h(\boldsymbol{x}) - j_1^{\mathrm{T}}(t)]\mathcal{J}_t p(t, \boldsymbol{x}) = 0 \, . \tag{3.14}$$

Proof. Since the single coordinate x was a passive parameter in the proof of theorem 1, playing no role in the essential identity (3.7), it can be expanded into a *set* of coordinates denoted by $\boldsymbol{x}$ in the statement of theorem 2, and the latter is proved by the same formal steps as was theorem 1. □

This higher-dimensional analogue of theorem 1 may be valuable in its own right, but we shall concentrate on the first theorem in what follows.

4. Two applications

The basic results of this paper are theorems 1 and 2, stated and proved in the preceding section, and their implications for wave equations with exact spreading solutions. The latter will be illustrated in this section with some equations arising in mathematical physics. Consider first the Klein–Gordon equation in Minkowski spacetime

$$[\partial_t^2 - \Delta + m^2]\phi = 0 \tag{4.1}$$

where Δ is the Laplacian and m is a constant. If we express the Laplacian in spherical coordinates and put $\phi(r, t, \theta, \varphi) = r\phi(r, t)Y_{lm}(\theta, \varphi)$, where the Y_{lm} are the usual spherical harmonics, (4.1) is equivalent to the set of equations

$$\left[\partial_t^2 - \partial_r^2 + m^2 + \frac{l(l+1)}{r^2}\right]\phi(r, t) = 0 \qquad l \in Z^+ \, . \tag{4.2}$$

Clearly (4.2) is an example of an equation to which the corollary to theorem 1 is applicable, with $j_1^{\mathrm{T}}(t)$ replaced by $g(t)$ and $h(r)$ replaced by $j_1^{\mathrm{S}}(r)$, where in this case $g(t) = -m^2$ and $j_1^{\mathrm{S}}(r) = -l(l+1)/r^2$. Thus we obtain analogues of (3.1) and (3.2) that are precisely the non-spreading equation (2.28) and the spreading equation (1.5), respectively. Since the general solution of (1.5) is

$$p(r, t) = \phi_+(r, t) + \phi_-(r, t) = \int_{-\infty}^{\infty} \mathrm{d}k \; g_+(k)\mathrm{e}^{\mathrm{i}[\omega(k)t - kx]} + \int_{-\infty}^{\infty} \mathrm{d}k \; g_-(k)\mathrm{e}^{\mathrm{i}[\omega(k)t + kx]} \tag{4.3}$$

where $\omega(k) = (k^2 + m^2)^{1/2}$, it follows from the corollary to theorem 1 that the general solution of (4.2) is the spreading wave

$$\phi(r, t) = \sum_{n=0}^{l} \frac{(-2)^{l-n}(l+n)!}{n!(l-n)!x^n} \partial_x^{l-n}[\phi_+(r, t) + \phi_-(r, t)] \, . \tag{4.4}$$

Passing the differentiation operator through the k-integration allows us to express $\phi(r, t, \theta, \varphi)$, the general solution of (4.1), in terms of integrals over Bessel polynomials, i.e. as a sum with respect to l and m, with arbitrary constant coefficients, of

$$\phi = Y_{lm}(\theta, \varphi) \int_{-\infty}^{\infty} \mathrm{d}k \; \mathrm{e}^{\mathrm{i}\omega(k)t}(2\mathrm{i})^l \left[g_+(k)\mathrm{e}^{-\mathrm{i}kx} B_l\left(\frac{1}{\mathrm{i}kx}\right) + (-1)^l g_-(k)\mathrm{e}^{\mathrm{i}kx} B_l\left(\frac{1}{-\mathrm{i}kx}\right)\right] . \tag{4.5}$$

This is a representation of the general solution of (4.2) that is exact in the same sense that (1.7) provides an exact solution of (1.5), and we see from (4.4) that the solution of the $l = 0$ mode of (4.1) serves as a potential for the $l > 0$ modes of (4.1).

For our other example we pass to a highly specific problem in cosmology that actually prompted the development of the general technique that is our main result in this paper. The Robertson–Walker spacetime

$$ds^2 = H(t)^2\{-dt^2 + dr^2 + f^2(r)d\Omega^2\} \tag{4.6}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ and

$$f(r) = \begin{cases} \sin r & K = +1 \text{ (closed)} \\ r & K = 0 \text{ (flat)} \\ \sinh r & K = -1 \text{ (open)} \end{cases} \tag{4.7}$$

is the most general homogeneous and isotropic spacetime, and is the starting point for a great variety of cosmological investigations. Its perturbations have been extensively studied, particularly its density perturbations, as they play an important role in studies of galaxy formation in the early universe. A systematic formalism for the analysis of cosmological perturbations was recently promoted by Stewart [17], and reviewed and applied in some detail by Mukhanov *et al* [18]. A minor modification of that formalism was suggested by Bombelli *et al* [19] in the context of a discussion of isentropic perfect fluid perturbations. In [18] and [19] it was independently derived that the density perturbations $\phi(r, t, \theta, \varphi)$ of an isentropic perfect fluid background spacetime are governed by a single scalar wave equation that is equivalent to the set of equations

$$\left[\partial_t^2 - c_S^2(t)\partial_r^2 + V(t) + \frac{l(l+1)c_S^2(t)}{f(r)^2}\right]\phi(r,t) = 0 \qquad l \in Z^+ \tag{4.8}$$

where $\phi(r, t, \theta, \varphi) = \phi(r, t)Y_{lm}$, and $c_S^2(t)$ and $V(t)$ are determined by the background spacetime (4.7) through the Einstein field equations. The reader is referred to [18] or [19] for the details, as our concern here is only with the particular scalar wave equations to which (4.8) reduces for some special choices of $H(t)$ and K in (4.7). Specifically, if we choose $H(t)$ and $f(r)$ to have the same functional form, i.e.

$$H(z) = f(z) = \begin{cases} \sin z & K = +1 \text{ (closed)} \\ z & K = 0 \text{ (flat)} \\ \sinh z & K = -1 \text{ (open)} \end{cases} \tag{4.9}$$

then (4.8) reduces to

$$\left\{\partial_y^2 - \partial_r^2 + \left[-\frac{2}{[f(\sqrt{3}y)/\sqrt{3}]^2} + |K| + \frac{l(l+1)}{f^2(r)}\right]\right\}\phi = 0 \tag{4.10}$$

where $y = t/\sqrt{3}$. The corresponding background pressure p and density μ are given by $\mu = 3p = H^{-2}(t)$, so these are the physically important 'radiation dominated' universes often used by cosmologists. What is of importance to us is the fact that (4.9) is exactly solvable by our methods in all three cases, i.e. $K = 1, K = 0, K = -1$.

If we first consider the case $K = 0$, $f(z) = z$, (4.10) reduces to (3.10), with t replaced by y, and $q = 1$. Thus the general solution in this case is given in terms of the progressing waves (3.11) with $q = 1$. Passing to the case $K = 1$, $f(z) = \sin z$, we see that (4.10) reduces to the equation

$$\left[\partial_y^2 - \partial_r^2 - \frac{2}{[\sin(\sqrt{3}y)/\sqrt{3}]^2} + 1 + \frac{l(l+1)}{\sin^2 r}\right]\phi = 0 . \tag{4.11}$$

But this is exactly solvable with a spreading solution that is obtained by consecutive applications of theorem 1 and its corollary. We begin by exactly solving the spreading equation (1.5) with $m = 1$. We then note that (4.11) with $l = 0$ can be exactly solved with a spreading wave by applying theorem 1, since $2/[\sin(\sqrt{3}y)/\sqrt{3}]^2$ is the second case of (2.25), with $t = y$, $l = 1$, and $d = \sqrt{3}$, while 1 is independent of y. We can then solve (4.11) itself by the application of the corollary to theorem 1, since $-l(l+1)/\sin^2 r$ is the second case of (2.23) with $x = r$, while $1 + 2/[\sin(\sqrt{3}y)/\sqrt{3}]^2$ is independent of r. The result is

$$\phi_\pm = \mathcal{J}_r\left(\{j_k^S(r)\}\right)\mathcal{J}_t\left(\{j_k^T(y)\}\right)(\rho_+ + \rho_-) \tag{4.12}$$

where

$$\rho_\pm = \int_{-\infty}^{\infty} \mathrm{d}k\; g_\pm(k)\mathrm{e}^{\mathrm{i}[\omega(k)t \mp kx]} \tag{4.13}$$

with $\omega(k) = (1 + k^2)^{1/2}$, and $\mathcal{J}_r$ and $\mathcal{J}_y$ calculated as usual from the substitution sequences corresponding to the second cases of (2.23) and (2.25). Once again the differentiation operators can be passed through the integration with respect to k to yield representations analogous to (4.5); however, the Bessel polynomials will be replaced by rational functions in sines and cosines of r and $\sqrt{3}y$, as appropriate. The third case, with $K = -1$ and $f(z) = \sinh z$, is similar, with hyperbolic functions in place of the circular ones.

There are in fact other background cosmological spacetimes for which (4.8) is exactly sovable by means of progressing waves, as in the first of our three cases, but most of them are of little interest because the corresponding pressures and densities are physically unrealistic. We are actively looking for other physically realistic cosmological backgrounds for which (4.8) can be exactly and generally solved by spreading waves through the application of theorem 1, as in the second and third cases just considered.

Acknowledgment

This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[2] Hayes W D 1970 Kinematic wave theory *Proc. R. Soc.* A **320** 209
[3] Hayes W D 1973 Group velocity and nonlinear dispersive wave propagation *Proc. R. Soc.* **332** 199
[4] Friedlander F G 1946 Simple progressive solutions to the wave equation *Proc. Camb. Phil. Soc.* **43** 360
[5] Courant R and Hilbert D 1961 *Methods of Mathematical Physics* vol 2 (Berlin: Springer)

[6] Friedlander F G 1975 *The Wave Equation on a Curved Spacetime* (Cambridge: Cambridge University Press)
[7] Kundt W and Newman E T 1968 Hyperbolic differential equations in two dimensions *J. Math. Phys* **9** 2193
[8] Torrence R J 1987 Linear wave equations as motions on a Toda lattice *J. Phys. A: Math. Gen.* **20** 91
[9] Eckhoff K S 1984 On dispersion for linear waves in nonuniform media *SIAM J. Appl. Math.* **44** 1092
[10] Hillion P 1992 Nondispersive solutions of the Klein–Gordon equation *J. Math. Phys.* **33** 1817
[11] Toda M 1989 *Theory of Nonlinear Lattices* (Berlin: Springer) 2nd edn
[12] Leznov A N and Saveliev M V *Group-Theoretical Methods for Integration of Nonlinear Dynamical Systems* (Basel: Birkhäuser)
[13] Couch W E and Torrence R J 1986 A class of wave equations with progressive wave solutions of finite order *Phys. Lett.* **117A** 270
[14] Torrence R J 1988 Simple Toda lattice motions and their linear wave equations *J. Phys. A: Math. Gen.* **21** 3205
[15] Krall H L and Frink O 1949 A new class of orthogonal polynomials: the Bessel polynomials *Trans. Am. Math. Soc.* **65** 100
[16] Torrence R J 1990 Self-adjoint acoustic equations with progressing wave solutions *J. Phys. A: Math. Gen.* **23** 4107
[17] Stewart J M 1990 Perturbations of Friedmann–Robertson–Walker cosmological models *Class. Quantum Grav.* **7** 1169
[18] Mukhanov V F, Feldman H A, and Brandenberger R H 1992 Theory of cosmological perturbations *Phys. Rep.* **215** 203
[19] Bombelli L, Couch W E, and Torrence R J 1992 Isentropic perfect fluid perturbations of cosmological spacetimes in Stewart's variables, submitted for publication